GOVERNMENT OF ANDHRA PRADESH

COMMISSIONERATE OF COLLEGIATE EDUCATION

SOLID GEOMETRY

THE PLANE

MATHEMATICS

GOGULAMUDI.SYAM PRASAD REDDY,
M.Sc.,M.Phil.,B.Ed.,SET

P.R.GOVT.COLLEGE(A):KAKINADA
Email ID :syam.g.reddy@gmail.com

Learn more at http:

THE PLANE

Learning Objectives

Students will be able to recognize the following properties of the Plane:

- Identify points, rays and segments using words and symbols.
- Knowing the general equation of a plane.
- Understand the transformation of the equation to the plane into normal form.
- Identify angles between the two planes.
- Identify the condition of parallelism.
- Identify the condition of perpendicularity.

THE PLANE

Def: A Plane is a surface such that if any two points are taken on it, the line joining them lies wholly on the surface.

The general equation of first degree in x, y, z always represents a plane.

Proof :

Let $ax + by + cz + d = 0$, $a^2 + b^2 + c^2 \neq 0$, be the first degree equation in x, y, z ……. (1)

If we have to show that (1) represents the equation to the plane , we prove that every point on the line joining any two points on (1) also lies on the locus (1).

Let $P(x_1, y_1, z_1)$ and $Q(x_2,y_2,z_2)$ be any two points on the locus represented by (1), then

$ax_1 + by_1 + cz_1 + d = 0$ ……(2)

$ax_2 + by_2 + cz_2 + d = 0$ ……..(3)

Let R be any point on the line segment joining the points P and Q. Suppose R divides PQ in th

then $R = \left(\dfrac{\lambda x_2 + x_1}{\lambda + 1}, \dfrac{\lambda y_2 + y_1}{\lambda + 1}, \dfrac{\lambda z_2 + z_1}{\lambda + 1}\right)$, $\lambda+1 \neq 0$.

We have to show that R lies on the locus (1) for all values of λ.

Substituting the coordinates of R in the LHS of (1), we get

$$a\frac{(\lambda x_2+x_1)}{\lambda+1}+b\frac{(\lambda y_2+y_1)}{\lambda+1}+c\frac{(\lambda z_2+z_1)}{\lambda+1}+d$$

$$=\frac{a(\lambda x_2+x_1)+b(\lambda y_2+y_1)+c(\lambda z_2+z_1)+d(\lambda+1)}{\lambda+1}$$

$$=\frac{\lambda(\mathrm{ax_2+by_2+cz_2})+(\mathrm{ax_1+by_1+cz_1+d})}{\lambda+1}$$

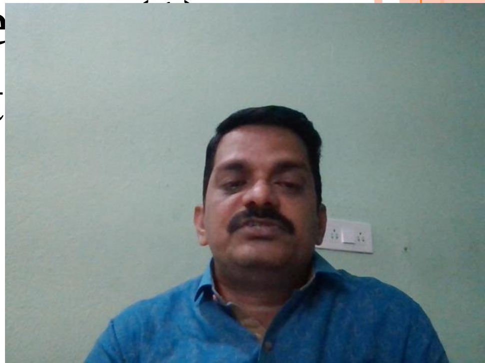
$$=\frac{\lambda(0)+0}{\lambda+1}$$

$$=0$$

Which shows that R lies on the locus (1).

Since R is an arbitrary point on the line joining P and Q, every point on PQ lie

Therefore the equation ax + by + cz + d = 0 , $a^2 + b^2 + c^2 \neq 0$ always represent



TRANSFORMATION OF THE EQUATION TO THE PLANE INTO NORMAL FORM

Let the equation to the plane be $ax + by + cz + d = 0$, $a^2 + b^2 + c^2 \neq 0$ …..(1)

We can take $d \geq 0$ or $d \leq 0$.

$$ax + by + cz + d = 0 \Leftrightarrow ax + by + cz = -d.$$

Dividing $\sqrt{a^2 + b^2 + c^2}$, we get , we get

$$\frac{-a}{\sqrt{a^2+b^2+c^2}}x + \frac{-b}{\sqrt{a^2+b^2+c^2}}y + \frac{-c}{\sqrt{a^2+b^2+c^2}}z = \frac{d}{\sqrt{a^2+b^2+c^2}} \qquad \ldots\ldots\ldots(2)$$

Or $$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = -\frac{d}{\sqrt{a^2+b^2+c^2}} \qquad \ldots\ldots.(3)$$

This is of the form $lx + my + nz = p$ ($p \geq 0$)

Where $l = \pm\frac{a}{\sqrt{\sum a^2}}$, $m = \pm\frac{b}{\sqrt{\sum b^2}}$, $n = \pm\frac{c}{\sqrt{\sum c^2}}$, $p = \mp\frac{d}{\sqrt{\sum a^2}}$

Therefore the normal form the equation to the plane (1) is

$$\pm\frac{ax}{\sqrt{\sum a^2}} \pm \frac{by}{\sqrt{\sum b^2}} \pm \frac{cz}{\sqrt{\sum c^2}} = \mp\frac{d}{\sqrt{\sum a^2}} \quad (d \leq 0 \text{ or } d \geq 0)$$

Theorem: If the equation $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ represents the same plane, then $a_1 : b_1 : c_1 : d_1 = a_2 : b_2 : c_2 : d_2$.

Proof: Given equations are $a_1x + b_1y + c_1z + d_1 = 0$ …..(1)

$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{….(2)}$$

$\therefore$ (a_1, b_1, c_1), (a_2, b_2, c_2) are d.rs. of normal's to the same plane.

Since the normal's are either equal (coincident) or parallel.

We have $a_1 : a_2 = b_1 : b_2 = c_1 : c_2 = \lambda$ (say) $(\lambda \neq 0)$ or $(a_1, b_1, c_1) = \lambda\,(a_2, b_2, c_2)$.

Let (x_1, y_1, z_1) be any point in the plane represented by (1) and (2).

$\therefore d_1 = -(\, a_1x_1 + b_1y_1 + c_1z_1\,)$

$= -\,(a_1, b_1, c_1).(x_1, y_1, z_1)$

$= -\lambda\,(a_2, b_2, c_2).\,(x_1, y_1, z_1)$

$= -\lambda\,(a_2x + b_2y + c_2z)$

$= \lambda d_2$

$\therefore$ $a_1 : b_1 : c_1 : d_1 = a_2 : b_2 : c_2 : d_2$. Or $a_1 : a_2 = b_1 : b_2 = c_1 : c_2 = d_1 : d_2$.

ANGLES BETWEEN TWO PLANES

Definition: Angles between two planes are equal to the angles between their normals.

Let the equations to the plane be $a_1x + b_1y + c_1z + d_1 = 0$ …..(1)

$a_2x + b_2y + c_2z + d_2 = 0$ ….(2)

Dc's of the normal to (1)= $m_1 = \left(\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right)$

and Dc's of the normal to (2) = $m_2 = \left(\frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$

Let θ be one of the angles between the planes.

$\therefore$ θ = one of the angles between the normal's m_1 , m_2

$$= \cos^{-1}\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

The other angle between the planes is 180° - θ.

CONDITION OF PARALLELISM

Planes are parallel $\Rightarrow \theta = 0^\circ$ or $180^\circ \Rightarrow \pm 1 = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

$\Rightarrow (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) = (a_1a_2 + b_1b_2 + c_1c_2)^2$

$\Rightarrow a_1^2b_2^2 + a_1^2c_2^2 + b_1^2a_2^2 + b_1^2c_2^2 + c_1^2a_2^2 + c_1^2b_2^2 - 2a_1a_2b_1b_2 - 2b_1b_2c_1c_2 - 2c_1c_2a_1a_2 = 0$

$\Rightarrow (a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 = 0$

$\Rightarrow a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 = c_1a_2 - c_2a_1 = 0$

$\Rightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2$

OR Planes are parallel $\Rightarrow$ their normal's are parallel

$\Rightarrow$ d.rs of normal's are proportional

$\Rightarrow a_1 : a_2 = b_1 : b_2 = c_1 : c_2$

CONDITION OF PERPENDICULARITY

Planes are perpendicular $\Rightarrow \theta = 90^\circ \Rightarrow$

OR Planes are perpendicular $\Rightarrow$ their normal's are perpendicular

$$\Rightarrow (a_1, b_1, c_1).(a_2, b_2, c_2) = 0$$

$$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Note. 1. The equations $a_1x + b_1y + c_1z + d_1 = 0$, $a_1x + b_1y + c_1z + d_2 = 0$ represents a pair of parallel planes.

2. A plane parallel to $ax + by + cz + d = 0$ is $ax + by + cz = k$, where k is an unknown real number.

The equation of the plane through the point (x_1, y_1, c_1) and parallel to the plane

$cz + d = 0$ is $ax + by + cz = ax_1 + by_1 + cz_1$.

Definition:

If a plane π intersects the coordinate axes at (a, 0, 0), (0, b, 0), (0, 0, c) then a, b, c are respectively called the x-intercept, the y-intercepts, the z-intercepts of the plane π.

If the plane $lx + my + nz = p$ intersects the x-axis at (a, 0, 0), then its x- intercept = $a = \frac{p}{l}$

Similarly its y-intercept = $b = \frac{p}{m}$, its z-intercept = $c = \frac{p}{n}$

Note. If $abc \neq 0$ and $ax + by + cz + d = 0$ is a plane, then then its x- intercept = $a = \frac{-d}{a}$

Similarly its y-intercept = $b = \frac{-d}{b}$, its z-intercept = $c = \frac{-d}{c}$.

THEOREM. **Equation to the plane making intercepts a, b, c on the coordinates axes is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.**

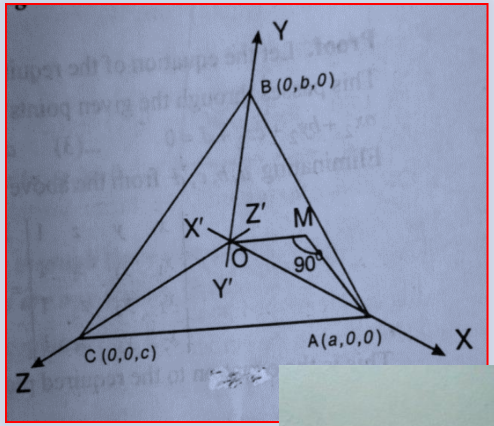
Proof. Let π be the plane making intercepts a, b, c on the coordinate axis.

Let A = (a, 0, 0), B = (0, b, 0) and C = (0, 0, c).

$\therefore abc \neq 0$. Clearly A, B, C are non-collinear.

Let the equation to the plane π in the normal form be $lx + my + nz = p$ ….. (1).

Let M be the foot of the perpendicular from O to π and let (l, m, n) be the Dc's of OM.



Let OM = p = Projection of OA on OM = al

Similarly p = bm, and p = cn.

$\therefore$ from(1) , equation to the plane π is $\frac{p}{a}x+\frac{p}{b}y+\frac{p}{c}z=p \Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Note. Equation to the plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

This is called the intercept form of the equation to the plane and this plane does

through the origin.


Thank you